ON PRODUCTS OF ELEMENTARILY INDIVISIBLE STRUCTURES

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ABSTRACT. We say a structure $\mathcal{M}$ in a first-order language $\mathcal{L}$ is *indivisible* if for every coloring of its universe M in two colors, there is a monochromatic substructure $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{M}' \cong \mathcal{M}$. Additionally, we say that $\mathcal{M}$ is *symmetrically indivisible* if $\mathcal{M}'$ can be chosen to be *symmetrically embedded* in $\mathcal{M}$ (that is, every automorphism of $\mathcal{M}'$ can be extended to an automorphism of $\mathcal{M}$). Similarly, we say that $\mathcal{M}$ is *elementarily indivisible* if $\mathcal{M}'$ can be chosen to be an elementary substructure. We define new products of structures in a relational language. We use these products to give recipes for construction of elementarily indivisible structures which are not transitive and elementarily indivisible structures which are not symmetrically indivisible, answering two questions presented by A. Hasson, M. Kojman and A. Onshuus.

1. INTRODUCTION

The notion of indivisibility of relational first-order structures and metric spaces is well studied in Ramsey theory. ([DLPS07],[EZS93], [EZS94] and [KR86] are just a few examples of the extensive study in this area.) Recall that a structure $\mathcal{M}$ in a relational first-order language is indivisible, if for every coloring of its universe M in two colors, there is a monochromatic substructure $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{M}' \cong \mathcal{M}$. Rado's random graph, the ordered set of natural numbers and the ordered set of rational numbers are just a few of the many examples. Weakenings of this notions have also been studied (see [Sau14]). A known extensively studied strengthening of this notion is the pigeonhole property (see [BCD00], [BD99]). For an extensive survey on indivisibility see [Fra00, Appendix A].

In [GK11], several induced Ramsey theorems for graphs were strengthened to a "symmetrized" version, in which the induced monochromatic subgraph satisfies that all members of a prescribed set of its partial isomorphisms extend to automorphisms of the colored graph. In [HKO11], following [GK11], a new strengthening of the notion of indivisibility was introduced:

Definition 1.1. We say a substructure $\mathcal{N} \subseteq \mathcal{M}$ is *symmetrically embedded* in $\mathcal{M}$ if every automorphism of $\mathcal{N}$ extends to an automorphism of $\mathcal{M}$.

We say that $\mathcal{M}$ is *symmetrically indivisible* if for every coloring of M in two colors, there is a monochromatic $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{M}'$ is isomorphic to $\mathcal{M}$ and $\mathcal{M}'$ is symmetrically embedded in $\mathcal{M}$.

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In [HKO11], several examples of symmetrically indivisible structures were investigated. Examples include the random graph ([GK11]), the ordered rational numbers, the ordered natural numbers, the universal n-hypergraph.

In the last section of [HKO11], another strengthening of the notion of indivisibility was introduced:

Definition 1.2. we say that $\mathcal{M}$ is *elementarily indivisible* if for every coloring of M in two colors, there is a monochromatic $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{M}'$ is isomorphic to $\mathcal{M}$ and $\mathcal{M}'$ is an elementary substructure of $\mathcal{M}$.

Classic examples for this notion, as given in [HKO11], are the random graph and the ordered rational numbers. A classic example of a symmetrically indivisible structure which is not elementarily indivisible is the ordered natural numbers, since every singleton is $\emptyset$-definable. (In fact, there is no proper elementary substructures of $\langle \omega, < \rangle$.)

In view of the above example, indivisibility should be viewed as a property of the pair $(\mathcal{M}, \mathcal{L})$ of a structure and the language in which it is given. Elementary indivisibility seems to be the right analogous property of the structure only (i.e., independent of its language). This statement is given a precise meaning in Lemma 2.18.

In [HKO11], The following questions were asked regarding the properties of elementarily indivisible structures, as well as the relation between this notion and the notion of symmetric indivisibility:

Question 1. Does elementary indivisibility imply symmetric indivisibility?

Question 2. Is every elementarily indivisible structure homogeneous?

Question 3. Is there a rigid elementarily indivisible structure?

In the literature the precise definition of homogeneity tends to vary; for example in [Mac11], a structure is said to be homogeneous if it is what we call ultrahomogeneous. Here we follow the conventions of [Hod93] and [Mar02], as presented in Definitions 1.5 and 1.6.

To quote [DLPS07] in a similar context, "The uncountable case is different as the indivisibility property may fail badly". In view of this, since the dawn of mankind (i.e. all the study mentioned above), indivisibility of first-order structures has been mostly studied in the countable context, since in the uncountable case set theoretic phenomena come into play. We note that while all results mentioned in this paper hold under the restriction to countable structures, in fact the countability assumption is superfluous.

In this paper, we investigate a construction we call the lexicographic product $\mathcal{M}[\mathcal{N}]$ of two relational structures $\mathcal{M}$ and $\mathcal{N}$, presented in Definition 1.7. We note this construction is very similar to the "composition" defined in [HKO11] and it generalizes the lexicographic order and the lexicographic product of graphs, as known in graph theory. In Section 2, We show that if $\mathcal{M}$ and $\mathcal{N}$ both admit quantifier elimination and every two singletons in $\mathcal{M}$ satisfy the same first-order formulas (i.e. the theory of $\mathcal{M}$ is *transitive* in the sense of Definition 1.11 below), then $\mathcal{M}[\mathcal{N}]^s$ admits quantifier elimination as well. We use this result to show that if $\mathcal{M}$ and $\mathcal{N}$ are both elementarily indivisible, then so are $\mathcal{M}[\mathcal{N}]$ and $\mathcal{M}[\mathcal{N}]^s$.

We further generalize the quantifier elimination result to a generalized product construction we introduce in Definition 1.8.

Applying the results mentioned above, in Section 3 we give general constructions of elementarily indivisible structures which are not transitive and in Section 4 of elementarily indivisible structures which are not symmetrically indivisible, answering Questions 1 and 2 negatively. Question 3 remains open.

1.1. **Preliminaries.**

Definition 1.3. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures and $B \subseteq M$, we say that $f : B \to \mathcal{N}$ is a *partial elementary* map if $\mathcal{M} \models \varphi(\bar{b}) \iff \mathcal{N} \models \varphi(f(\bar{b}))$ for all $\mathcal{L}$-formulas φ and all finite sequences $\bar{b}$ from B.

If $B = M$ we just say f is an *elementary embedding*.

A substructure $\mathcal{M} \subseteq \mathcal{N}$ is an *elementary substructure* if the inclusion map ι is an elementary embedding, in which case we denote $\mathcal{M} \preceq \mathcal{N}$.

Definition 1.4. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures and $B \subseteq M$, we say that $f : B \to \mathcal{N}$ is a *partial isomorphism* if $\mathcal{M} \models \varphi(\bar{b}) \iff \mathcal{N} \models \varphi(f(\bar{b}))$ for all *quantifier-free* (or equivalently, atomic) $\mathcal{L}$-formulas φ and all finite sequences $\bar{b}$ from B.

Definition 1.5. We say a structure $\mathcal{M}$ is *homogeneous* if whenever $A \subset M$ with $|A| < |M|$ and $f : A \to M$ is a partial elementary map, there is an automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma \restriction A = f$.

Definition 1.6. We say a structure $\mathcal{M}$ is *ultrahomogeneous* if whenever $A \subset M$ with $|A| < |M|$ and $f : A \to M$ is a partial isomorphism, there is an automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma \restriction A = f$.

In [HKO11], a construction very similar to the following was introduced. We note that while our construction is slightly different, in fact, in the context of binary relational languages these two definitions coincide.

Definition 1.7. Let $\mathcal{M}$, $\mathcal{N}$ be structures in a relational language, $\mathcal{L}$. The *lexicographic product* $\mathcal{M}[\mathcal{N}]$ is the $\mathcal{L}$-structure whose universe is $M \times N$ where for every n-ary relation $R \in \mathcal{L}$ we set

$$R^{\mathcal{M}[\mathcal{N}]} :=$$
$$\left\{ \big((a_1,b_1),\ldots,(a_n,b_n)\big) \;\middle|\; \textstyle\bigwedge_{1\le i,j\le n} a_i = a_j \quad \text{and} \quad \mathcal{N} \models R(b_1,\ldots,b_n) \right\} \;\cup$$
$$\left\{ \big((a_1,b_1),\ldots,(a_n,b_n)\big) \;\middle|\; \textstyle\bigvee_{1\le i\neq j\le n} a_i \neq a_j \quad \text{and} \quad \mathcal{M} \models R(a_1,\ldots,a_n) \right\}.$$

Let $\mathcal{M}[\mathcal{N}]^s$ be $\mathcal{M}[\mathcal{N}]$ expanded by a binary relation s interpreted as

$$\left\{ \big((a_1,b_1),(a_2,b_2)\big) \in (M \times N)^2 \;\middle|\; a_1 = a_2 \right\}.$$

For the purposes of this paper, we generalize the definition above to the following.

Definition 1.8. Let $\mathcal{M}$, $\{\mathcal{N}_a\}_{a\in M}$ be structures in a relational language, $\mathcal{L}$. The *generalized product* $\mathcal{M}[\mathcal{N}_a]_{a\in M}$ is the $\mathcal{L}$-structure whose universe is $\bigcup_{a\in M}\{a\} \times N_a$ where for every n-ary relation $R \in \mathcal{L}$ we set

$$R^{\mathcal{M}[\mathcal{N}_a]_{a\in M}} :=$$
$$\left\{ \big((a,b_1),\ldots,(a,b_n)\big) \;\middle|\; \mathcal{N}_a \models R(b_1,\ldots,b_n) \right\} \;\cup$$
$$\left\{ \big((a_1,b_1),\ldots,(a_n,b_n)\big) \;\middle|\; \textstyle\bigvee_{1\le i\neq j\le n} a_i \neq a_j \quad \text{and} \quad \mathcal{M} \models R(a_1,\ldots,a_n) \right\}.$$

Let $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ be $\mathcal{M}[\mathcal{N}_a]_{a\in M}$ expanded by a binary relation s interpreted as

$$\left\{ \big((a,b_1),(a,b_2)\big) \;\middle|\; b_1,b_2 \in N_a \right\}.$$

Note that if there is a fixed $\mathcal{N}$ such that $\mathcal{N}_a = \mathcal{N}$ for all $a \in M$, then this definition coincides with $\mathcal{M}[\mathcal{N}]$ and $\mathcal{M}[\mathcal{N}]^s$.

Remark 1.9. Notice that the interpretation of unary predicates in the product does not depend on their interpretation in $\mathcal{M}$, i.e. for a unary predicate $U \in \mathcal{L}$,

$$\mathcal{M}[\mathcal{N}_a]_{a\in M} \models U((a,b)) \iff \mathcal{N}_a \models U(b).$$

Remark 1.10. Notice that if $\mathcal{M}$, $\{\mathcal{N}_a\}_{a\in M}$ are structures in a relational language $\mathcal{L}$ and $a \in M$, then the substructure $\{a\} \times \mathcal{N}_a$ is isomorphic to $\mathcal{N}_a$.

Definition 1.11. We say a theory T is *transitive* if for every $\phi(x)$ in one free variable, either $T_1 \models \forall x\, \phi(x)$ or $T_1 \models \forall x\, \neg\phi(x)$ (i.e. $|S_1(T_1)| = 1$).

Lemma 1.12. $\mathrm{Th}(\mathcal{M})$ is transitive for every elementarily indivisible $\mathcal{L}$-structure $\mathcal{M}$.

Proof. If $\mathrm{Th}(\mathcal{M})$ is not transitive, then there is an $\mathcal{L}$-formula in one free variable $\phi(x)$ such that $\mathrm{Th}(\mathcal{M}) \not\models \forall x\, \phi(x)$ and $\mathrm{Th}(\mathcal{M}) \not\models \forall x\, \neg\phi(x)$. By completeness of $\mathrm{Th}(\mathcal{M})$, $\mathrm{Th}(\mathcal{M}) \models \exists x\, \neg\phi(x)$ and $\mathrm{Th}(\mathcal{M}) \models \exists x\, \phi(x)$. Define a coloring $c : M \to \{\mathrm{red}, \mathrm{blue}\}$ as follows:

$$c(x) := \begin{cases} \text{blue} & \text{if } \mathcal{M} \models \phi(x) \\ \text{red} & \text{if } \mathcal{M} \models \neg\phi(x). \end{cases}$$

It is clear that no c-monochromatic substructure is elementary. $\square$

Note that obviously if $\mathcal{M}$ is a transitive structure (i.e. for every $a, b \in M$ there is an automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma(a) = b$), then $\mathrm{Th}(\mathcal{M})$ is transitive, but the converse is not necessarily true – in fact, in Section 3 we will see examples of elementarily indivisible structures which are not transitive. Having said that, we do have:

Remark 1.13. If $\mathcal{M}$ is homogeneous, then $\mathcal{M}$ is transitive iff $\mathrm{Th}(\mathcal{M})$ is transitive.

Corollary 1.14. Every homogeneous elementarily indivisible structure is transitive. $\square$

2. Elimination of quantifiers

In this paper we stick to the definition of quantifier elimination presented in [Mar02]:

Definition 2.1. We say that an $\mathcal{L}$-theory T admits *quantifier elimination* (QE) if for every $\mathcal{L}$-formula ϕ there is a quantifier-free formula ψ such that

$$T \models \phi \leftrightarrow \psi.$$

We say an $\mathcal{L}$-structure $\mathcal{M}$ admits QE if $\mathrm{Th}(\mathcal{M})$ admits QE.

Remark 2.2. If T admits QE and $\mathcal{N}, \mathcal{M} \models T$, $\mathcal{N} \subseteq \mathcal{M}$ then $\mathcal{N} \prec \mathcal{M}$. So if $\mathcal{M}$ is indivisible and admits QE, then it is elementarily indivisible.

Furthermore: this remark can be extended to any infinitary logic. For that, we extend the definition of QE to $\mathcal{L}_{\kappa,\lambda}$ in a natural way:

Definition 2.3. We say that an $\mathcal{L}_{\kappa,\lambda}$-theory T admits *quantifier elimination* (QE) if for every $\mathcal{L}_{\kappa,\lambda}$-formula ϕ there is a quantifier-free $\mathcal{L}_{\kappa,\lambda}$-formula ψ such that

$$T \models \phi \leftrightarrow \psi.$$

We say an $\mathcal{L}$-structure $\mathcal{M}$ admits $\mathcal{L}_{\kappa,\lambda}$-QE if its $\mathcal{L}_{\kappa,\lambda}$-theory admits QE.

Remark 2.4. It is an easy exercise to verify that every ultrahomogeneous structure $\mathcal{M}$ in a relational language admits $\mathcal{L}_{|\operatorname{Th}(\mathcal{M})|^+,|\mathcal{M}|}$-QE, which, in turn, implies that every embedding is elementary. So we have that every indivisible ultrahomogeneous structure is elementarily indivisible.

Throughout this section, we use the following abbreviations:

Notation 2.5.

- $\bar{v} := (v_1, \ldots, v_n)$ is an n-tuple of variables.
- $\overline{(a, b)} := \big((a_1, b_1), \ldots, (a_n, b_n)\big)$ is an n-tuple of elements in the product (generalized or not).
- Whenever $\overline{(a, b)} = \big((a_1, b_1), \ldots, (a_n, b_n)\big)$, we denote $\bar{a} := (a_1, \ldots, a_n)$ and $\bar{b} := (b_1, \ldots, b_n)$.

Notice that whenever $\bar{v}$ and $\overline{(a, b)}$ appear together, they are of the same length.

2.1. **The elimination.** In this subsection we prove the following theorem which is the main result of the section.

Theorem 2.6. Let $\mathcal{L}$ be a relational language and let T_1, T_2 be $\mathcal{L}$-theories, not necessarily complete. If T_1 and T_2 both admit QE and T_1 is transitive then there is an $\mathcal{L} \cup \{s\}$-theory T (not necessarily complete) admitting QE, such that $\mathcal{M}[\mathcal{N}_a]^s_{a \in M} \models T$ whenever $\mathcal{M} \models T_1$ and $\{\mathcal{N}_a\}_{a \in M} \models T_2$.

In particular, if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures both admitting QE and $\operatorname{Th}(\mathcal{M})$ is transitive then $\mathcal{M}[\mathcal{N}]^s$ admits QE.

Before proving this theorem, we note that the requirement of transitivity is necessary and provide a simple example in which $\mathcal{M}$ and $\mathcal{N}$ both admit QE, but $\mathcal{M}[\mathcal{N}]^s$ does not:

Example 2.7. Let $\mathcal{L} := \{ R, A, B \}$ where R is a binary relation and A, B are unary predicates. Let $\mathcal{M}$ be an $\mathcal{L}$-structure satisfying:

- $\left|A^{\mathcal{M}}\right| = 1$, $\left|B^{\mathcal{M}}\right| = \aleph_0$
- $A^{\mathcal{M}} \cap B^{\mathcal{M}} = \emptyset$.
- $R^{\mathcal{M}} := \{ (a, b) \mid a \in A,\ b \in B \}$.

Let $\mathcal{N}$ be an $\mathcal{L}$-structure with a countably infinite universe interpreting all relations in $\mathcal{L}$ as empty. Then $\mathcal{M}$ and $\mathcal{N}$ both admit QE but $\mathcal{M}[\mathcal{N}]^s$ does not admit QE.

Proof. Obviously $\mathcal{N}$ admits QE. To show $\mathcal{M}$ admits QE, since R is quantifier-free $\emptyset$-definable in $\mathcal{M}$, it suffices to show $\mathcal{M} \restriction \{A, B\}$ admits QE (where $\mathcal{M} \restriction \{A, B\}$ is the restriction of $\mathcal{M}$ to the language $\{A, B\}$), but this is again obvious.

To show $\mathcal{M}[\mathcal{N}]^s$ does not admit QE, by Remark 1.9, $\mathcal{M}[\mathcal{N}]^s \models U((x, y)) \iff \mathcal{N} \models U(y)$ for every unary predicate $U \in \mathcal{L}$. Since $\mathcal{N}$ interprets all relations in $\mathcal{L}$ as empty, $\mathcal{M}[\mathcal{N}]^s$ interprets all unary predicates as empty. Thus every quantifier-free formula in one variable is equivalent to either "$x = x$" or "$x \neq x$". Let $\phi(x) :=$

$\exists y\, R(x,y)$. Notice that $\mathcal{M}[\mathcal{N}]^s \models \phi((a,c))$ for $a \in A^{\mathcal{M}}$ and $\mathcal{M}[\mathcal{N}]^s \not\models \phi((b,c))$ for $b \in B^{\mathcal{M}}$. So $\phi(x)$ is neither equivalent to "$x = x$" nor to "$x \neq x$" and thus $\mathcal{M}[\mathcal{N}]^s$ does not admit QE.

□

We continue with a few definitions and lemmas needed for the proof of Theorem 2.6. First, we introduce a notation for a manipulation on formulas that we will use several times in this subsection:

Notation 2.8. Let ϕ be an $\mathcal{L}$-formula. We denote $\widetilde{\phi}$ the $\mathcal{L} \cup \{s\}$-formula obtained from ϕ by replacing the equality symbol '=' with s, namely:

- If ϕ is atomic of the form $R(\bar{v})$ for $R \in \mathcal{L}$, then $\widetilde{\phi} := \phi$.
- If ϕ is atomic of the form "$x = y$", then $\widetilde{\phi} := s(x,y)$.
- If ϕ is of the form $\alpha * \beta$ where $* \in \{ \wedge, \vee, \rightarrow \}$, then $\widetilde{\phi} := \widetilde{\alpha} * \widetilde{\beta}$.
- If ϕ is of the form $\neg\beta$, then $\widetilde{\phi} := \neg\widetilde{\beta}$.
- If ϕ is of the form $*x\, \beta$ where $* \in \{ \forall, \exists \}$ then $\widetilde{\phi} := *x\, \widetilde{\beta}$.

Lemma 2.9. If $\phi(\bar{v})$ is a quantifier-free $\mathcal{L}$-formula, $a \in M,\ b_1, \dots, b_n \in N_a$, then

$$\mathcal{M}[\mathcal{N}_a]_{a\in M} \models \phi\left((a,b_1), \dots, (a,b_n)\right) \iff \mathcal{N}_a \models \phi(b_1, \dots, b_n).$$

Proof. Define $e_a : \mathcal{N}_a \to \mathcal{M}[\mathcal{N}_a]_{a\in M}$ by $e_a(b) := (a,b)$. By definition of $\mathcal{M}[\mathcal{N}_a]_{a\in M}$ this is an $\mathcal{L}$-embedding and thus the claim follows. □

Definition 2.10. Let $\phi(\bar{v})$ be an $\mathcal{L}$-formula, $\overline{(a,b)} \in \mathcal{M}[\mathcal{N}_a]_{a\in M}$. We say $\overline{(a,b)}$ is an *admissible assignment* for ϕ when for every $R \in \mathcal{L}$, if $R(v_{i_1}, \dots, v_{i_k})$ occurs in ϕ, then $\bigvee_{1\le l<m\le k} a_{i_l} \neq a_{i_m}$.

Lemma 2.11. If $\phi(\bar{v})$ is a quantifier-free $\mathcal{L}$-formula and $\overline{(a,b)} \in \mathcal{M}[\mathcal{N}_a]_{a\in M}$ is an admissible assignment for ϕ, then

$$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \widetilde{\phi}\left(\overline{(a,b)}\right) \iff \mathcal{M} \models \phi(\bar{a}).$$

In particular, If $\phi(\bar{v})$ is a quantifier-free $\mathcal{L}$-formula such that the equality symbol does not occur in ϕ, then

$$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \phi\left(\overline{(a,b)}\right) \Leftrightarrow \mathcal{M} \models \phi\left(\bar{a}\right).$$

Proof.

- If ϕ is of the form "$v_1 = v_2$", this follows by definition of s.
- If ϕ is of the form $R(v_{i_1}, \dots, v_{i_k})$, since $\overline{(a,b)}$ is an admissible assignment, by definition of $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$,

$$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \phi\left(\overline{(a,b)}\right) \iff \mathcal{M} \models \phi\left(\bar{a}\right)$$

  and $\widetilde{\phi} = \phi$.
- For a general quantifier-free ϕ the claim follows by induction on the complexity of ϕ.

□

Definition 2.12. A formula $\phi(\bar{v})$ is called a *complete equality diagram* if it is a consistent conjunction of formulas of the form "$x = y$" and "$x \neq y$" such that for every $i \le i, j \le n$, either $\phi\left(\bar{v}\right) \vdash v_i = v_j$ or $\phi\left(\bar{v}\right) \vdash v_i \neq v_j$.

Lemma 2.13. Let T be a transitive theory. For every quantifier-free $\mathcal{L}$-formula $\varphi(\bar{v})$ there is a quantifier-free $\mathcal{L}\cup\{s\}$-formula $\varphi'(\bar{v})$ such that if $\mathcal{M}\models T, \{\mathcal{N}_a\}_{a\in M}$ are $\mathcal{L}$-structures, $\overline{(a,b)}\in\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$, then

$$\mathcal{M}\models\varphi(\bar{a}) \iff \mathcal{M}[\mathcal{N}_a]^s_{a\in M}\models\varphi'\left(\overline{(a,b)}\right).$$

Proof. Let $\{\psi_j\}_{j\in J}$ be all complete equality diagrams on $\bar{v}$. Notice that

$$\vdash\varphi(\bar{v})\leftrightarrow\left(\varphi(\bar{v})\wedge\bigvee_{j\in J}\psi_j(\bar{v})\right)\leftrightarrow\bigvee_{j\in J}\left(\varphi(\bar{v})\wedge\psi_j(\bar{v})\right).$$

So by taking disjunctions, it suffices to find a quantifier-free $\mathcal{L}\cup\{s\}$-formula $\varphi'(\bar{v})$ such that for every $\mathcal{L}$-structures $\mathcal{M},\{\mathcal{N}_a\}_{a\in M}$ such that $\mathcal{M}\models T$ and for every $\overline{(a,b)}\in\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$,

$$\mathcal{M}\models\varphi(\bar{a})\wedge\psi(\bar{a}) \iff \mathcal{M}[\mathcal{N}_a]^s_{a\in M}\models\varphi'\left(\overline{(a,b)}\right)$$

where ψ is a complete equality diagram.

Next, for every v_j, v_k such that $j<k$ and $\psi(\bar{v})\vdash v_j=v_k$, we can replace every occurrence of v_k with v_j, so we may assume $\psi(\bar{v})=\bigwedge_{1\le j<k\le n}v_j\neq v_k$. Secondly, since T is transitive, every formula of the form $R(x,\dots,x)$ is equivalent either to "$x=x$" or to "$x\neq x$", so we may assume there are no such occurrences in φ. Let $\widetilde{\psi},\widetilde{\varphi}$ be the formulas obtained from ψ,φ respectively, by replacing '$=$' with s. We claim that for every $\mathcal{L}$-structures $\mathcal{M}\models T,\{\mathcal{N}_a\}_{a\in M}$ and $\overline{(a,b)}\in\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$,

$$\mathcal{M}\models\varphi(\bar{a})\wedge\psi(\bar{a}) \iff \mathcal{M}[\mathcal{N}_a]^s_{a\in M}\models\widetilde{\varphi}\left(\overline{(a,b)}\right)\wedge\widetilde{\psi}\left(\overline{(a,b)}\right).$$

Indeed: by definition, $\mathcal{M}\models\psi(\bar{a}) \iff \mathcal{M}[\mathcal{N}_a]^s_{a\in M}\models\widetilde{\psi}\left(\overline{(a,b)}\right)$. Assuming $\mathcal{M}\models\psi(\bar{a})$, since there are no occurrences of the form $R(x,\dots,x)$ in φ, $\overline{(a,b)}$ is an admissible assignment for φ. So by Lemma 2.11,

$$\mathcal{M}\models\varphi(\bar{a}) \iff \mathcal{M}[\mathcal{N}_a]^s_{a\in M}\models\widetilde{\varphi}\left(\overline{(a,b)}\right).$$

□

Before continuing to the main proof – one last definition, that stand at the core of the proof of Theorem 2.6:

Definition 2.14. An $\{s\}$-formula $\phi(\bar{v})$ is called a *complete s-diagram* if it is a conjunction of formulas of the form $s(x,y)$ or $\neg s(x,y)$ such that for every $i\le i,j\le n$, either $\phi(v_1,\dots,v_n)\vdash s(v_i,v_j)$ or $\phi(\bar{v})\vdash\neg s(v_i,v_j)$ and ϕ is consistent with s being an equivalence relation.

Notice that ϕ is a complete s-diagram iff it is of the form $\widetilde{\psi}$ for some complete equality diagram ψ.

Proof of Theorem 2.6. We provide a technical proof, noting that this proof is in fact constructive, using the elimination of quantifiers from T_1 and T_2.

Let $\phi=\exists w\bigwedge_{i\in I}\theta_i(\bar{v},w)$ such that $\{\theta_i\}_{i\in I}$ are atomic and negated atomic formulas. We need to find a quantifier-free $\mathcal{L}\cup\{s\}$-formula φ such that for every $\mathcal{M}\models T_1$ and $\{\mathcal{N}_a\}_{a\in M}\models T_2$,

$$\mathcal{M}[\mathcal{N}_a]^s_{a\in M}\models\phi(\bar{v})\leftrightarrow\varphi(\bar{v}).$$

First, since $\vdash\exists w\big(\chi(\bar{v},w)\wedge\theta(\bar{v})\big)\leftrightarrow\exists w\big(\chi(\bar{v},w)\big)\wedge\theta(\bar{v})$ we may assume that w occurs in θ_i for all $i\in I$.

In order to proceed with the proof we will use complete s-diagrams, in a way similar to the way complete equality diagrams were used in the proof of Lemma 2.13:

Let T_{equiv} be the $\{s\}$-theory stating that s is an equivalence relation and let $\{\widetilde{\psi}_j\}_{j\in J}$ be all the complete s-diagrams on $\bar{v}, w$. There are finitely many such and

$$T_{equiv} \models \exists w \bigwedge_{i\in I} \theta_i(\bar{v}, w) \leftrightarrow \exists w\big(\bigwedge_{i\in I} \theta_i(\bar{v}, w) \wedge \bigvee_{j\in J} \widetilde{\psi}_j(\bar{v}, w)\big)$$

$$\leftrightarrow \exists w \bigvee_{j\in J} \big(\widetilde{\psi}_j(\bar{v}, w) \wedge \bigwedge_{i\in I} \theta_i(\bar{v}, w)\big)$$

$$\leftrightarrow \bigvee_{j\in J} \exists w\big(\widetilde{\psi}_j(\bar{v}, w) \wedge \bigwedge_{i\in I} \theta_i(\bar{v}, w)\big).$$

Since $\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models T_{equiv}$ for every $\mathcal{M}$ and $\{\mathcal{N}_a\}_{a\in M}$, we may assume ϕ is of the form $\exists w\big(\widetilde{\psi}(\bar{v}, w) \wedge \bigwedge_{i\in I} \theta_i(\bar{v}, w)\big)$ where $\widetilde{\psi}$ is a complete s-diagram, θ_i are atomic and negated atomic formulas such that w occurs in each θ_i.

Next, let

$$I_2 := \left\{\, i \in I \;\middle|\; \widetilde{\psi} \vdash s(v, w) \text{ for all } v \text{ occuring in } \theta_i \,\right\}$$

$$I_1 := \left\{\, i \in I \;\middle|\; \widetilde{\psi} \vdash \neg s(v, w) \text{ for some } v \text{ occuring in } \theta_i \,\right\} = I \setminus I_2$$

and separate $\bar{v}$ to $\bar{v}^1, \bar{v}^2$, where $\bar{v}^2$ are the variables occurring in $\bigwedge_{i\in I_2} \theta_i(\bar{v}, w)$ and $\bar{v}^1$ the ones not occurring. So ϕ is of the form

$$\exists w \left(\widetilde{\psi}(\bar{v}^1, \bar{v}^2, w) \wedge \bigwedge_{i\in I_1} \theta_i(\bar{v}^1, \bar{v}^2, w) \wedge \bigwedge_{i\in I_2} \theta_i(\bar{v}^2, w)\right)$$

where $\widetilde{\psi}$ is a complete s-diagram. We may further assume '=' and s do not occur in $\bigwedge_{i\in I_1} \theta_i(\bar{v}^1, \bar{v}^2, w)$, for such an occurrence would be either superfluous with respect to $\widetilde{\psi}$ or inconsistent with $\widetilde{\psi}$.

If $\bar{v}^1 = (v^1_1, \ldots, v^1_{n_1}), \bar{v}^2 = (v^2_1, \ldots, v^2_{n_2})$, let

$$\begin{aligned} \bar{a}^1 &= (a^1_1, \ldots, a^1_{n_1}),\ \bar{b}^1 = (b^1_1, \ldots, b^1_{n_1}) \\ \bar{a}^2 &= (a^2_1, \ldots, a^1_{n_2}),\ \bar{b}^2 = (b^1_1, \ldots, b^1_{n_2}) \end{aligned}$$

and denote

$$\overline{(a,b)}^1 := \big((a^1_1, b^1_1), \ldots, (a^1_{n_1}, b^1_{n_1})\big) \qquad \overline{(a,b)}^2 := \big((a^2_1, b^2_1), \ldots, (a^2_{n_2}, b^2_{n_2})\big)\,.$$

Claim. The following are equivalent:

(1)

$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models$

$$\exists w \left(\widetilde{\psi}\left(\overline{(a,b)}^1, \overline{(a,b)}^2, w\right) \wedge \bigwedge_{i\in I_1} \theta_i \left(\overline{(a,b)}^1, \overline{(a,b)}^2, w\right) \wedge \bigwedge_{i\in I_2} \theta_i \left(\overline{(a,b)}^2, w\right)\right)$$

(2) There is an $a \in M$ such that: $a^2_{j_2} = a$ for all $1 \leq j \leq n_2$,

$$\mathcal{M} \models \exists w \left(\psi(\bar{a}^1, \bar{a}^2, w) \wedge \bigwedge_{i \in I_1} \theta_i(\bar{a}^1, \bar{a}^2, w) \right) \text{ and } \mathcal{N}_a \models \exists w \left(\bigwedge_{i \in I_2} \theta_i(\bar{b}^2, w) \right)$$

Proof of Claim.
($\Rightarrow$) Let $c \in M$ and $d \in N_c$ such that

$$\mathcal{M}[\mathcal{N}_a]^s_{a \in M} \models$$
$$\widetilde{\psi}\left(\overline{(a,b)}^1, \overline{(a,b)}^2, (c,d)\right) \wedge \bigwedge_{i \in I_1} \theta_i \left(\overline{(a,b)}^1, \overline{(a,b)}^2, (c,d)\right) \wedge \bigwedge_{i \in I_2} \theta_i \left(\overline{(a,b)}^2, (c,d)\right).$$

By definition, $\mathcal{M} \models \psi(\bar{a}^1, \bar{a}^2, c)$, and since $\psi(\bar{a}^1, \bar{a}^2, c)$ implies that $\overline{(a,b)}^1, \overline{(a,b)}^2, (c,d)$ is an admissible assignment for $\bigwedge_{i \in I_1} \theta_i(\bar{v}^1, \bar{v}^2, w)$, by Lemma 2.11,

$$\mathcal{M} \models \bigwedge_{i \in I_1} \theta_i(\bar{a}^1, \bar{a}^2, c).$$

Furthermore, by the definition of I_2,

$$\psi\left(\bar{a}^1, \bar{a}^2, c\right) \vdash \left(\bigwedge_{1 \leq j \leq n_2} a^2_j = c \right) \wedge \left(\bigwedge_{1 \leq j,k \leq n_2} a^2_j = a^2_k \right).$$

So letting $a := c$, in fact

$$\mathcal{M}[\mathcal{N}_a]^s_{a \in M} \models \bigwedge_{i \in I_1} \theta_i \left((a, b^2_1), \ldots, (a, b^2_{n_2}), (a,d)\right)$$

so by Lemma 2.9,

$$\mathcal{N}_a \models \bigwedge_{i \in I_2} \theta_i(\bar{b}^2, d).$$

($\Leftarrow$) Let $a \in M$ be such that for all $1 \leq j \leq n_2$, $a^2_{j_2} = a$, and let $c \in M$ and $d \in N_a$ be such that

$$\mathcal{M} \models \psi\left(\bar{a}^1, \bar{a}^2, c\right) \wedge \bigwedge_{i \in I_1} \theta_i \left(\bar{a}^1, \bar{a}^2, c\right) \text{ and } \mathcal{N}_a \models \bigwedge_{i \in I_2} \theta_i \left(\bar{b}^2, d\right).$$

Since $\psi\left(\bar{a}^1, \bar{a}^2, c\right) \vdash \bigwedge_{1 \leq j \leq n_2} a^2_j = c$, in fact $c = a$, so $d \in N_c$ and

$$\mathcal{N}_c \models \bigwedge_{i \in I_2} \theta_i \left(\bar{b}^2, d\right),$$

thus by Lemma 2.9,

$$\mathcal{M}[\mathcal{N}_a]^s_{a \in M} \models \bigwedge_{i \in I_2} \theta_i \left(\overline{(a,b)}^2, (c,d)\right).$$

Since $\psi\left(\bar{a}^1, \bar{a}^2, c\right)$ implies that $\overline{(a,b)}^1, \overline{(a,b)}^2, (c,d)$ is an admissible assignment for ϕ_1, by Lemma 2.11,

$$\mathcal{M}[\mathcal{N}_a]^s_{a \in M} \models \widetilde{\psi}\left(\overline{(a,b)}^1, \overline{(a,b)}^2, (c,d)\right) \wedge \bigwedge_{i \in I_1} \theta_i \left(\overline{(a,b)}^1, \overline{(a,b)}^2, (c,d)\right).$$

$\square$ Claim

Assuming $\mathcal{M} \models T_1$ and $\{\mathcal{N}_a\}_{a\in M} \models T_2$, by QE of T_1 and T_2, let $\varphi_1(\bar{v}), \varphi_2(\bar{v})$ be quantifier-free $\mathcal{L}$-formulas such that

$$T_1 \models \exists w \left(\widetilde{\psi}\left(\bar{v}^1, \bar{v}^2, w\right) \wedge \bigwedge_{i\in I_1} \theta_i(\bar{v}^1, \bar{v}^2, w) \right) \leftrightarrow \varphi_1\left(\bar{v}^1, \bar{v}^2\right) \quad \text{and}$$

$$T_2 \models \exists w \left(\bigwedge_{i\in I_2} \theta_i(\bar{v}^2, w) \right) \leftrightarrow \varphi_2\left(\bar{v}^2\right).$$

So (2) above is equivalent to:

(3) There is an $a \in M$ such that: $a^2_{j_2} = a$ for all $1 \le j \le n_2$,
$$\mathcal{M} \models \varphi_1\left(\bar{a}^1, \bar{a}^2\right) \text{ and } \mathcal{N}_a \models \varphi_2\left(\bar{b}^2\right).$$

By Lemmas 2.9 and 2.13, there is an $\mathcal{L}$-formula φ_1' such that (3) above is equivalent to:

(4) $\bigwedge_{1\le j,k\le n_2} a^2_j = a^2_k$ and
$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \varphi_1'\left(\overline{(a,b)}^1, \overline{(a,b)}^2\right)$ and $\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \varphi_2\left(\overline{(a,b)}^2\right)$,
which is equivalent to

(5)
$$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models$$
$$\left(\bigwedge_{1\le j,k\le n_2} s\left((a^2_j, b^2_j), (a^2_k, b^2_k)\right) \right) \wedge \varphi_1'\left(\overline{(a,b)}^1, \overline{(a,b)}^2\right) \wedge \varphi_2\left(\overline{(a,b)}^2\right).$$

Setting

$$\varphi\left(\bar{v}^1, \bar{v}^2\right) := \left(\bigwedge_{1\le j,k\le n_2} s\left(v^2_j, v^2_k\right) \right) \wedge \varphi_1'\left(\bar{v}^1, \bar{v}^2\right) \wedge \varphi_2\left(\bar{v}^2\right),$$

we get that for every $\overline{(a,b)}^1 \in \left(\mathcal{M}[\mathcal{N}_a]^s_{a\in M}\right)^{n_1}$ and $\overline{(a,b)}^2 \in \left(\mathcal{M}[\mathcal{N}_a]^s_{a\in M}\right)^{n_2}$:

$$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \phi\left(\overline{(a,b)}^1, \overline{(a,b)}^2\right) \iff \mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \varphi\left(\overline{(a,b)}^1, \overline{(a,b)}^2\right).$$

So

$$\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models \phi\left(\bar{v}^1, \bar{v}^2\right) \leftrightarrow \varphi\left(\bar{v}^1, \bar{v}^2\right)$$

and φ is quantifier-free.

Let φ_ϕ be the quantifier-free $\mathcal{L} \cup \{s\}$-formula obtained from ϕ by the above process. Let T be the logical closure (all the logical consequences) of

$$T_{equiv} \cup \left\{ \phi \leftrightarrow \varphi_\phi \;\middle|\; \phi \text{ is of the form } \exists w \left(\widetilde{\psi}(\bar{v}, w) \wedge \bigwedge_{i\in I} \theta_i(\bar{v}, w) \right) \right\}.$$

T admits QE and by the above process, $\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models T$ for every $\mathcal{M} \models T_1$, $\{\mathcal{N}_a\}_{a\in M} \models T_2$. $\square$

Note that in the proof above, transitivity of T is used to get from (3) to (4) as φ_1 can include occurrences of the form $R(x, \dots, x)$ that would be interpreted in the product differently in each copy of $\mathcal{N}_a$, and in general we cannot use Lemma 2.13 if $\mathrm{Th}(\mathcal{M})$ is not transitive.

We note that if T_1 and T_2 are complete, so is T and thus:

Corollary 2.15. If $\mathcal{M}_1 \equiv \mathcal{M}_2$ and $\{ N_a \mid a \in M_1 \cup M_2 \}$ are pairwise elementarily equivalent then $\mathcal{M}_1[\mathcal{N}_a]_{a\in M_1} \equiv \mathcal{M}_2[\mathcal{N}_a]_{a\in M_2}$.

We leave it as an exercise to show that s is necessary; i.e. find $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ (even elementarily indivisible), such that $\mathcal{M}$ and $\mathcal{N}$ both admit QE but $\mathcal{M}[\mathcal{N}]$ does not (not even model complete).

2.2. **Application to elementary indivisibility.**

In this subsection, we provide an immediate application of Theorem 2.6 to elementary indivisibility, mainly proving that the lexicographic product of two elementarily indivisible structures is elementarily indivisible. Here we only use the result of QE for $\mathcal{M}[\mathcal{N}]^s$, though in the following sections the full power of Theorem 2.6 regarding the generalized product will be needed.

Definition 2.16. Let $\mathcal{M}$ and $\mathcal{M}'$ be structures with the same universe M, not necessarily in the same language.

We say $\mathcal{M}'$ is a *language reduct* of $\mathcal{M}$ if $\mathcal{M}' = \mathcal{M} \restriction \mathcal{L}_0$ for some $\mathcal{L}_0 \subseteq \mathcal{L}$.

We say $\mathcal{M}'$ is a *definitional reduct* of $\mathcal{M}$ if every $\emptyset$-definable relation in $\mathcal{M}'$ is $\emptyset$-definable in $\mathcal{M}$.

Notation 2.17. Let $\widehat{\mathcal{L}}$ be an expansion of $\mathcal{L}$ such that for each $\mathcal{L}$ -formula $\phi(\bar{v})$ with n free variables, we add an n-ary relation R_ϕ (we denote by $\phi_R(\bar{v})$ the formula that defined R).

For any $\mathcal{L}$-structure $\mathcal{M}$, we define $\widehat{\mathcal{M}}$ an $\widehat{\mathcal{L}}$ -structure whose universe is the universe of $\mathcal{M}$, and for every n-ary relation symbol $R \in \widehat{\mathcal{L}}$ we set

$$R^{\widehat{\mathcal{M}}} = \begin{cases} R^{\mathcal{M}} & \text{if } R \in \mathcal{L} \\ \{ \bar{a} \in \mathcal{M}^n \mid \mathcal{M} \models \phi_R(\bar{a}) \} & \text{if } R \in \widehat{\mathcal{L}} \setminus \mathcal{L} \end{cases}$$

We call $\widehat{\mathcal{M}}$ the Morleyzation of $\mathcal{M}$. It is well-known and an easy exercise to verify that $\widehat{\mathcal{M}}$ admits QE.

We note that while it is obvious that if $\mathcal{M}$ is (elementarily) indivisible and $\mathcal{M}'$ is a language reduct of $\mathcal{M}$, then $\mathcal{M}'$ is also (elementarily) indivisible, this is not true for definitional reducts. For example consider the ordered natural numbers $\langle \omega, < \rangle$. The following lemma implies this is not the case in the *elementarily* indivisible context. Because of this, and following [Mac11] and the extensive study done in the subject, we use *reduct* as an abbreviation for definitional reduct.

Lemma 2.18. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The following are equivalent:

1. $\mathcal{M}$ is elementarily indivisible.
2. $\widehat{\mathcal{M}}$ is indivisible.
3. $\widehat{\mathcal{M}}$ is elementarily indivisible.
4. every reduct of $\mathcal{M}$ is indivisible.
5. every reduct of $\mathcal{M}$ is elementarily indivisible.

Proof.

- (5)⇒(4)⇒(2) is obvious, since $\widehat{\mathcal{M}}$ is a reduct of $\mathcal{M}$.
- (2)⇒(3) is by quantifier elimination of the Morleyzation, and model completeness (Remark 2.2).
- (3)⇒(1) is due to elementary indivisibility respecting language reducts.

- (1)⇒(5) Let $\mathcal{M}'$ be a reduct of $\mathcal{M}$ in a language $\mathcal{L}'$. Let $c : M \to \{0,1\}$ be a coloring and let $\mathcal{N}$ be a monochromatic elementary substructure isomorphic to $\mathcal{M}$ with universe $N \subseteq M$. We will show the induced $\mathcal{L}'$-substructure of $\mathcal{M}'$ on N is an elementary substructure isomorphic to $\mathcal{M}'$. Since $\mathcal{M}'$ is a reduct of $\mathcal{M}$, for every $\mathcal{L}'$-formula ϕ, there is an $\mathcal{L}$-formula φ_ϕ such that $\mathcal{M}' \models \phi(\bar{a}) \iff \mathcal{M} \models \varphi(\bar{a})$ for every $\bar{a} \in M$. In particular, for every $R \in \mathcal{L}'$, there is an $\mathcal{L}$-formula φ_R such that $\mathcal{M}' \models R(\bar{a}) \iff \mathcal{M} \models \varphi_R(\bar{a})$.

  Let $\mathcal{N}'$ be the $\mathcal{L}'$-structure whose universe is N and for every $R \in \mathcal{L}'$
$$R^{\mathcal{N}'} := \{\, \bar{a} \mid \mathcal{N} \models \varphi_R(\bar{a}) \,\}.$$
  Since $\mathcal{N} \cong \mathcal{M}$, also $\mathcal{N}' \cong \mathcal{M}'$. Since $\mathcal{N} \prec \mathcal{M}$, for every $R \in \mathcal{L}'$ we have
$$\mathcal{N}' \models R(\bar{a}) \iff \mathcal{N} \models \varphi_R(\bar{a}) \iff \mathcal{M} \models \varphi_R(\bar{a}) \iff \mathcal{M}' \models R(\bar{a}),$$
  so, in fact, $\mathcal{N}'$ coincides with the induced $\mathcal{L}'$-substructure of $\mathcal{M}$ on N. But the above equivalence can also be achieved for $\mathcal{L}'$-formulas:
$$\mathcal{N}' \models \phi(\bar{a}) \iff \mathcal{N} \models \varphi_\phi(\bar{a}) \iff \mathcal{M} \models \varphi_\phi(\bar{a}) \iff \mathcal{M}' \models \phi(\bar{a}).$$
  So $\mathcal{N}'$ is an elementary substructure of $\mathcal{M}'$.

□

The following proposition is in fact almost identical to a part of [HKO11, Proposition 2.14], but for the sake of completeness we give a simple proof here.

Proposition 2.19. If $\mathcal{M}$ and $\mathcal{N}$ are both indivisible then so is $\mathcal{M}[\mathcal{N}]^s$

Proof. Let $c : \mathcal{M}[\mathcal{N}]^s \to \{0,1\}$ be a coloring of $\mathcal{M}[\mathcal{N}]^s$. So for each $a \in M$, c induces a coloring of $\{a\} \times \mathcal{N}$ and $\{a\} \times \mathcal{N} \cong \mathcal{N}$, so $\{a\} \times \mathcal{N}$ is indivisible. So for each $a \in M$ there is $\mathcal{N}(a) \subseteq \{a\} \times \mathcal{N}$ s.t. $|c[\mathcal{N}(a)]| = 1$ and $\mathcal{N}(a) \cong \mathcal{N}$. Now, let us define a coloring $C : \mathcal{M} \to \{\{0\},\{1\}\}$ as follow: $C(a) := c[\mathcal{N}(a)]$. From the previous statement it follows that C is well-defined. So C is a coloring of $\mathcal{M}$ and since $\mathcal{M}$ is indivisible, there is as C-monochromatic substructure $\mathcal{M}_0 \subseteq \mathcal{M}$ isomorphic to $\mathcal{M}$. Let $\mathcal{A} \subseteq \mathcal{M}[\mathcal{N}]^s$ be the substructure
$$\bigcup_{a \in M_0} \mathcal{N}(a).$$
By its construction, $\mathcal{A}$ is monochromatic in c. We next show that it is isomorphic to $\mathcal{M}[\mathcal{N}]^s$ and the proposition follows.

Let $f : \mathcal{M}_0 \xrightarrow{\cong} \mathcal{M}$ be an isomorphism and for every $a \in M_0$, let $g_a : \mathcal{N}(a) \xrightarrow{\cong} \mathcal{N}$ be an isomorphism. We define $F : \mathcal{A} \to \mathcal{M}[\mathcal{N}]^s$ by
$$F((a,b)) = (\ f(a),\ g_a((a,b))\ ).$$
We leave it to the reader to verify that F is indeed an isomorphism.

□

Theorem 2.20. If $\mathcal{M}$ and $\mathcal{N}$ are elementarily indivisible, then so are $\mathcal{M}[\mathcal{N}]$ and $\mathcal{M}[\mathcal{N}]^s$.

Proof. First note that $\mathcal{M}[\mathcal{N}]$ is a reduct of $\mathcal{M}[\mathcal{N}]^s$, so it suffices to show elementary indivisibility only for $\mathcal{M}[\mathcal{N}]^s$.

From the assumption and by Lemma 2.18, $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ are elementarily indivisible in $\widehat{\mathcal{L}}$, thus by Proposition 2.19, $\widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s$ is indivisible in $\widehat{\mathcal{L}}$. By Lemma 1.12, $\mathrm{Th}(\widehat{\mathcal{M}})$ is transitive and since $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ both admit QE in $\widehat{\mathcal{L}}$, by Theorem 2.6, $\widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s$ admits

QE. In conclusion, $\widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s$ is indivisible and admits QE, thus it is elementarily indivisible and $\mathcal{M}[\mathcal{N}]^s$ is a reduct of $\widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s$ to $\mathcal{L}$. $\square$

We observe that $\widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s$ is not the same structure as $\widehat{\mathcal{M}[\mathcal{N}]}^s$. Thus, Theorem 2.6 does not automatically imply QE relative to $\mathcal{M}$ and $\mathcal{N}$, i.e., the QE assumption in statement of the theorem cannot be dropped. The following proposition remedies this situation.

Proposition 2.21. Let $\mathcal{M},\mathcal{N}$ be structures in a relational language such that $\mathrm{Th}(\mathcal{M})$ is transitive.

If $\phi(\bar{v})$ is any $\mathcal{L}\cup\{s\}$-formula, then there are $\mathcal{L}$-formulas $\left\{ \varphi_1^j(\bar{v}), \varphi_2^j(\bar{v}) \right\}_{j=1}^k$ such that for every $\overline{(a,b)} \in \mathcal{M}[\mathcal{N}]^s$:

$$\mathcal{M}[\mathcal{N}]^s \models \phi\left(\overline{(a,b)}\right) \iff \bigvee_{j=1}^{k} \left(\mathcal{M} \models \varphi_1^j\left(\bar{a}\right) \wedge \mathcal{N} \models \varphi_2^j\left(\bar{b}\right)\right)$$

Proof. Since $\mathcal{M}[\mathcal{N}]^s = \widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s \restriction \mathcal{L}\cup\{s\}$ and since $\widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s$ admits QE, there is a quantifier-free $\widehat{\mathcal{L}}\cup\{s\}$-formula $\varphi(\bar{v})$ such that

$$\mathcal{M}[\mathcal{N}]^s \models \phi\left(\overline{(a,b)}\right) \iff \widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s \models \phi\left(\overline{(a,b)}\right) \iff \widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s \models \varphi\left(\overline{(a,b)}\right)$$

for every $\overline{(a,b)} \in \mathcal{M}[\mathcal{N}]^s$. By taking the disjunctive normal form (DNF) of $\varphi(\bar{v})$, conjuncting with the disjunction with all complete s-diagrams and using disjunctions, we may assume $\varphi(\bar{v})$ is of the form $\widetilde{\psi}(\bar{v}) \wedge \bigwedge_{i\in I}\theta_i(\bar{v})$ where θ_i are atomic and negated atomic formulas. As in the proof of Theorem 2.6, there are quantifier-free $\widehat{\mathcal{L}}$-formulas $\widehat{\varphi_1}(\bar{v})$ and $\widehat{\varphi_2}(\bar{v})$ such that

$$\widehat{\mathcal{M}}[\widehat{\mathcal{N}}]^s \models \varphi\left(\overline{(a,b)}\right) \iff \widehat{\mathcal{M}} \models \widehat{\varphi_1}(\bar{a}) \text{ and } \widehat{\mathcal{N}} \models \widehat{\varphi_2}(\bar{b})$$

for every $\overline{(a,b)} \in \mathcal{M}[\mathcal{N}]^s$. Since $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ are reducts of $\mathcal{M}$ and $\mathcal{N}$ respectively, there are $\mathcal{L}$-formulas $\varphi_1(\bar{v})$ and $\varphi_2(\bar{v})$ such that

$$\widehat{\mathcal{M}} \models \widehat{\varphi_1}(\bar{a}) \iff \mathcal{M} \models \varphi_1(\bar{a}) \text{ and } \widehat{\mathcal{N}} \models \widehat{\varphi_2}(\bar{b}) \iff \mathcal{N} \models \varphi_2(\bar{b})$$

for every $\overline{(a,b)} \in \mathcal{M}[\mathcal{N}]^s$. $\square$

3. The existence of non-transitive elementarily indivisible structures

In this section, we give a construction for non-transitive elementarily indivisible structures. Noting that every elementary indivisible homogeneous structure is transitive, this gives a negative answer to Question 2. In Subsection 3.1 we prove the main result of this section and in Subsection 3.2, we generalize this result by constructing elementarily indivisible structures with infinitely many orbits. The generalization will be used in Section 4.

3.1. **Two orbits.**

Definition 3.1. Let $\mathcal{L}$ be a relational language, an elementarily indivisible pair in $\mathcal{L}$ is a pair of elementarily indivisible $\mathcal{L}$-structures $\langle \mathcal{M}_0, \mathcal{M}_1 \rangle$ such that $\mathcal{M}_0 \prec \mathcal{M}_1$ and $\mathcal{M}_0 \ncong \mathcal{M}_1$.

The existence of such a pair is needed for our construction, and thus we hereby present a key example:

Example 3.2. Let $n \geq 2$ (it wouldn't harm to assume $n = 2$)

Let $\mathcal{L} = \{R_i\}_{i\in\omega}$ where all R_i are of arity n. Let $\mathcal{C}$ be the class of all finite $\mathcal{L}$-structures, satisfying:

(1) All R_i are:
- Symmetric, i.e. $R_i(v_1, \dots, v_n) \to R_i(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ for every $\sigma \in S_n$.
- Irreflexive, i.e. $\left(\bigvee_{1\leq j<k\leq n} v_j = v_k\right) \to \neg R_i(v_1, \dots, v_n)$.

and

(2) All R_i are disjoint, i.e. $R_{i_1}(v_1, \dots, v_n) \to \neg R_{i_2}(v_1, \dots, v_n)$ whenever $i_1 \neq i_2$.

$\mathcal{C}$ could be thought of as the class of all finite n-hypergraphs (for $n = 2$ this is simply graphs) with edges colored in ω colors. This is a Fraïssé class and let $\mathcal{M}_1$ be its Fraïssé limit. Explicitly $\mathcal{M}_1$ is the unique (up to isomorphism) $\mathcal{L}$-structure satisfying the property, in addition to (1),(2) above:

(A) For every finite $X \subset M_1$, $c : [X]^{n-1} \to \{-1\}\cup\omega$ there is a $v \in M_1$ satisfying that for every $x_1, \dots, x_{n-1} \in X$:
- If $i = c(\{a_1, \dots, a_{n-1}\}) \in \omega$ then $(a_1, \dots, a_{n-1}, v) \in R_i$.
- If $c(\{a_1, \dots, a_{n-1}\}) = -1$ then $\bigwedge_{j\in\omega}(a_1, \dots, a_{n-1}, v) \notin R_j$.

Let $\mathcal{D}$ be the class of all finite $\mathcal{L}$-structures, satisfying (1),(2) above and in addition:

(3) Completeness: $\left(\bigwedge_{1\leq j<k\leq n} v_j \neq v_k\right) \to \left(\bigvee_{i\in\omega} R_i(v_1, \dots, v_k)\right)$

$\mathcal{D}$ could be thought of as the class of all finite *complete* n-hypergraphs with edges colored in ω colors. Note that $\mathcal{D}$ is not an elementary class, but it is a Fraïssé subclass of $\mathcal{C}$. Let $\mathcal{M}_0$ be its Fraïssé limit. Explicitly $\mathcal{M}_0$ is the unique (up to isomorphism) $\mathcal{L}$-structure satisfying the following property, in addition to (1),(2),(3) above:

(A′) For every finite $X \subset M_0$, $c : [X]^{n-1} \to \omega$ there is a $v \in M_0$ satisfying that for every $x_1, \dots, x_{n-1} \in X$ that if $i = c(\{a_1, \dots, a_{n-1}\})$ then $(a_1, \dots, a_{n-1}, v) \in R_i$.

By universality, $\mathcal{M}_0$ embeds into $\mathcal{M}_1$, assume, without loss of generality, that $\mathcal{M}_0 \subset \mathcal{M}_1$. It is well known and easy to verify that $\mathcal{M}_1 \equiv \mathcal{M}_0$ and they admit QE, so $\mathcal{M}_0 \prec \mathcal{M}_1$ and since $\operatorname{age}(\mathcal{M}_0) \subsetneq \operatorname{age}(\mathcal{M}_1)$, $\mathcal{M}_0 \ncong \mathcal{M}_1$.

The elementary indivisibility of both $\mathcal{M}_0$ and $\mathcal{M}_1$ can be shown in either one of the following methods:

(1) Use the universal properties of $\mathcal{M}_0$ and $\mathcal{M}_1$. The proof of for $n = 2$ is given in [HKO11, Example 6.12] and for $n \geq 3$ the proof is exactly the same.

(2) Using the uniqueness properties of $\mathcal{M}_0$ and $\mathcal{M}_1$, the well known proof of *indivisibility* of the random graph (given in [Hen71, Corollary 1.5]) can be easily generalized to $\mathcal{M}_0$ and $\mathcal{M}_1$. Regarding *elementary indivisibility*: the theory of these structures is known to admit QE, thus elementary indivisibility follows from indivisibility, but from their definition as Fraïssé limits, they are ultrahomogeneous and likewise elementary indivisibility follows from indivisibility and Remark 2.4.

Lemma 3.3. For $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, we denote $\mathcal{M} \sim_e \mathcal{N}$ if both $\mathcal{M}$ can be elementarily embedded in $\mathcal{N}$ and vice-versa.

If $\mathcal{M} \sim_e \mathcal{N}$ then $\mathcal{M}$ is elementarily indivisible iff $\mathcal{N}$ is elementarily indivisible.

Proof. Because $\sim_e$ is an equivalence relation, it suffices to show one direction. Suppose $\mathcal{M}$ is elementarily indivisible and assume, without loss of generality, $\mathcal{M} \preceq \mathcal{N}$. Let $c : \mathcal{N} \to \{\text{red}, \text{blue}\}$ be a coloring of $\mathcal{N}$, so c naturally induces a coloring of $\mathcal{M}$. Since $\mathcal{M}$ is elementarily indivisible, there is a c-monochromatic $\mathcal{M}' \prec \mathcal{M}$ such that $\mathcal{M}' \cong \mathcal{M}$. Now, since $\mathcal{N}$ can be elementarily embedded in $\mathcal{M}$ and $\mathcal{M}' \cong \mathcal{M}$, in particular, there is some $\mathcal{N}_0 \prec \mathcal{M}'$ such that $\mathcal{N}_0 \cong \mathcal{N}$ and since $\mathcal{M}'$ is monochromatic, so is $\mathcal{N}_0$. □

Lemma 3.4. Let $\mathcal{M}$, $\{\mathcal{N}_a\}_{a\in M}$ be $\mathcal{L}$-structures. For every $(a_1, b_1), (a_2, b_2) \in \mathcal{M}[\mathcal{N}_a]^s_{a\in M}$, if there is an automorphism $\sigma \in \operatorname{Aut} \mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ such that $\sigma(a_1, b_1) = (a_2, b_2)$, then $\mathcal{N}_{a_1} \cong \mathcal{N}_{a_2}$.

Proof. σ sends s-equivalence classes to s-equivalence classes and $\{a\} \times \mathcal{N}_a$ is an s-equivalence class for every $\{a\} \in M$. Therefore $\sigma[\{a_1\} \times \mathcal{N}_{a_1}] = \{a_2\} \times \mathcal{N}_{a_2}$, so $\sigma \restriction \{a_1\} \times \mathcal{N}_{a_1} : \{a_1\} \times \mathcal{N}_{a_1} \to \{a_2\} \times \mathcal{N}_{a_2}$ is an isomorphism, but $\{a_1\} \times \mathcal{N}_{a_1} \cong \mathcal{N}_{a_1}$ and $\{a_2\} \times \mathcal{N}_{a_2} \cong \mathcal{N}_{a_2}$. □

Theorem 3.5. Let $\mathcal{M}$ be a transitive elementarily indivisible structure and $\langle \mathcal{N}_0, \mathcal{N}_1 \rangle$ an elementarily indivisible pair. Let $\mathcal{M}' \subseteq \mathcal{M}_1 \subset \mathcal{M}$ be such that $\mathcal{M}' \cong \mathcal{M}$ and $\mathcal{M}' \prec \mathcal{M}$ and let

$$\mathcal{N}_a := \begin{cases} \mathcal{N}_1 & \text{if } a \in \mathcal{M}_1 \\ \mathcal{N}_0 & \text{if } a \notin \mathcal{M}_1. \end{cases}$$

Then the generalized product $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ is elementarily indivisible and is not transitive.

Proof. To prove $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ is elementarily indivisible, we may assume that $\mathcal{M}, \mathcal{N}_0, \mathcal{N}_1$ all admit QE. If not, by looking at $\widehat{\mathcal{M}}, \widehat{\mathcal{N}_0}, \widehat{\mathcal{N}_1}$, the assumptions remain true and we can, assuming QE, prove that $\widehat{\mathcal{M}}[\widehat{\mathcal{N}_a}]^s_{a\in M}$ is elementarily indivisible, so $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ is also elementarily indivisible, as a reduct of such.

We will show that $\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \sim_e \mathcal{M}[\mathcal{N}_1]^s$. By Theorem 2.20, $\mathcal{M}[\mathcal{N}_1]^s$ is elementarily indivisible, thus by Lemma 3.3, $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ will also be elementarily indivisible.

Clearly $\mathcal{M}[\mathcal{N}_1]^s \cong \mathcal{M}'[\mathcal{N}_1]^s$ and $\mathcal{M}'[\mathcal{N}_1]^s \subset \mathcal{M}[\mathcal{N}_a]^s_{a\in M}$, so there is an embedding $e_1 : \mathcal{M}[\mathcal{N}_1]^s \hookrightarrow \mathcal{M}[\mathcal{N}_a]^s_{a\in M}$; On the other hand, clearly $\mathcal{M}[\mathcal{N}_a]^s_{a\in M} \subset \mathcal{M}[\mathcal{N}_1]^s$, so we have an embedding $e_2 : \mathcal{M}[\mathcal{N}_a]^s_{a\in M} \hookrightarrow \mathcal{M}[\mathcal{N}_1]^s$.

Now by QE of $\mathcal{M}$ and of $\mathcal{N}_0 \equiv \mathcal{N}_1$ and by Theorem 2.6, there is an $\mathcal{L} \cup \{s\}$-theory T admitting QE, such that $\mathcal{M}[\mathcal{N}_1]^s, \mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models T$. By QE of T, e_1 and e_2 are elementary embeddings. □

Corollary 3.6. There is an elementarily indivisible structure that is not transitive and not homogeneous.

Proof. Let $\mathcal{M}$ be any transitive elementarily indivisible structure (all the classic examples in the introduction are), let $\langle \mathcal{N}_0, \mathcal{N}_1 \rangle$ be an elementarily indivisible pair (the existence of such a pair is established in Example 3.2), let $\mathcal{M}_1 \subset \mathcal{M}$ such that there is some $\mathcal{M}' \subseteq \mathcal{M}_1$ satisfying $\mathcal{M}' \cong \mathcal{M}$ and $\mathcal{M}' \prec \mathcal{M}$ (by elementary indivisibility, there are $2^{\aleph_0}$ such but it does not harm to assume $\mathcal{M}_1$ is co-finite) and let

$$\mathcal{N}_a := \begin{cases} \mathcal{N}_1 & \text{if } a \in \mathcal{M}_1 \\ \mathcal{N}_0 & \text{if } a \notin \mathcal{M}_1. \end{cases}$$

By Theorem 3.5, $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ is elementarily indivisible and not transitive. By Corollary 1.14, $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ is not homogeneous. $\square$

3.2. **Infinitely many orbits.** In this subsection, we generalize the result from Subsection 3.1 and prove the existence of an elementarily indivisible structure with infinitely many orbits under its automorphism group. We will use such a structure in Section 4.

For the construction, we need an infinite set of elementarily indivisible structures satisfying the following.

Lemma 3.7. There is an infinite set of elementarily indivisible pairwise-non-isomorphic structures $\{\mathcal{A}_i\}_{i\in\omega}$, such that $\mathcal{A}_i \sim_e \mathcal{A}_j$ for all $i,j\in\omega$.

Proof. Let $\mathcal{M}$ be a transitive elementarily indivisible structure and $\langle \mathcal{N}_0, \mathcal{N}_1\rangle$ an elementarily indivisible pair. Without loss of generality, they all admit QE. Let $\mathcal{M} \supseteq \mathcal{M}_0 \supsetneq \mathcal{M}_1 \supsetneq \mathcal{M}_2 \supsetneq \dots$ be an infinite descending chain of substructures satisfying the following:

- $\mathcal{M}$ can be embedded into $\mathcal{M}_i$ for every $i\in\omega$.
- For every $0 \le i < j \le \omega$, either $\mathcal{M}_i \ncong \mathcal{M}_j$ or $\mathcal{M}\setminus\mathcal{M}_i \ncong \mathcal{M}\setminus\mathcal{M}_j$.

By induction and indivisibility of $\mathcal{M}$, given $\mathcal{M}_i$, there are many appropriate choices for $\mathcal{M}_{i+1}$ (though there is no harm in assuming $\mathcal{M}_0 = \mathcal{M}_1$ and $\mathcal{M}_{i+1}$ is just a co-finite substructure of $\mathcal{M}_i$).

For every $i\in\omega$ and $a\in M$, denote

$$\mathcal{N}^i_a := \begin{cases} \mathcal{N}_1 & \text{if } a\in\mathcal{M}_i \\ \mathcal{N}_0 & \text{if } a\notin\mathcal{M}_i \end{cases}$$

and let $\mathcal{A}_i := \mathcal{M}[\mathcal{N}^i_a]^s_{a\in M}$. Clearly $\mathcal{M}[\mathcal{N}_1]^s \supseteq \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$ and they are pairwise-non-isomorphic. Since $\mathcal{M}$ can be embedded into each $\mathcal{M}_i$, and each $\mathcal{A}_i$ embeds $\mathcal{M}_i[\mathcal{N}_1]^s$, it follows that $\mathcal{M}[\mathcal{N}_1]^s$ can be embedded into each $\mathcal{A}_i$. Each $\mathcal{A}_i$ can be embedded into $\mathcal{M}[\mathcal{N}_1]^s$ via the inclusion map. By Theorem 2.6 these embeddings are elementary, so $\mathcal{A}_i \sim_e \mathcal{M}[\mathcal{N}_1]^s$ for every $i\in\omega$. By Theorem 2.20 the latter is elementarily indivisible and thus by Lemma 3.3 so are all $\mathcal{A}_i$. $\square$

Theorem 3.8. Let $\{A_i\}_{i\in\omega}$ be as in Lemma 3.7 and let $\mathcal{M}$ be an elementarily indivisible structure. If $\{\mathcal{N}_a\}_{a\in M}$ is a collection of structures satisfying

$$\{\, \mathcal{N}_a \mid a\in M \,\} = \{\, \mathcal{A}_i \mid i\in\omega \,\}$$

(as sets), then $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ is elementarily indivisible and has infinitely many orbits. In particular, by Lemma 3.7, there is such a structure.

Proof. Without loss of generality, $T_1 := \mathrm{Th}(\mathcal{M})$ admits QE and there is an $\mathcal{L}$-theory T_2, admitting QE, such that $\mathcal{N}_a \models T_2$ for all $a\in M$. Let T be as guaranteed by Theorem 2.6. So $\mathcal{M}[\mathcal{N}]^s, \mathcal{M}[\mathcal{N}_a]^s_{a\in M} \models T$ and obviously $\mathcal{M}[\mathcal{A}_0]^s$ can be embedded into $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ and vice versa. By QE of T, these embeddings are elementary, so $\mathcal{M}[\mathcal{A}_0]^s \sim_e \mathcal{M}[\mathcal{N}_a]^s_{a\in M}$. By Theorem 2.20, $\mathcal{M}[\mathcal{A}_0]^s$ is elementarily indivisible and thus by Lemma 3.3 so is $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$.

Now, since, by choice of $\{\mathcal{N}_a\}_{a\in M}$, there are infinitely many pairwise non-isomorphic $\mathcal{N}_a$s, by Lemma 3.4, $\mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ has infinitely many orbits. $\square$

4. An elementarily non-symmetrically indivisible structure

In this section we will provide an negative answer to Question 1.

But first, we provide a simpler construction of an indivisible structure that is not symmetrically indivisible. This construction is given to provide the reader with intuition for the continuation of this section and will be generalized in Proposition 4.3. The quick reader may skip the following example.

Example 4.1. Let $\mathcal{L} = \{<\}$, let ω be the $\mathcal{L}$-structure of ordered natural numbers and let X a pure countably infinite set (letting $<^X = \emptyset$). Then $X[\omega]$ is indivisible but not symmetrically indivisible.

Proof. $X[\omega]$ is indivisible by Proposition 2.19. As for symmetric indivisibility – let $\{x_i\}_{i\in\omega}$ be an enumeration of X and $c : X[\omega] \to \{\text{red}, \text{blue}\}$ be the coloring defined as follows:

$$c(x_i, j) := \begin{cases} \text{red} & \text{if } j \leq i \\ \text{blue} & \text{if } j > i. \end{cases}$$

Every monochromatic red substructure will have only finite $<$-chains, and thus not isomorphic to $X[\omega]$. It is left to show that there is no monochromatic blue symmetrically embedded substructure isomorphic to $X[\omega]$. Assume towards contradiction $\mathcal{B}$ is such a structure and let $(x_{i_0}, j_0) \in \mathcal{B}$. Since $\mathcal{B} \cong X[\omega]^s$, $\mathcal{B}$ has infinitely many infinite $<$-chains and every chain is of the form $\mathcal{B} \cap (\{x_i\} \times \omega)$. So let $i_1 > j_0$ be such that $\mathcal{B} \cap (\{x_{i_1}\} \times \omega) \neq \emptyset$. Let $\sigma \in \mathrm{Aut}(\mathcal{B})$ be such that $\sigma[\mathcal{B} \cap (\{x_{i_0}\} \times \omega)] = \mathcal{B} \cap (\{x_{i_1}\} \times \omega)$ and let $(x_{i_1}, j_1) := \sigma(x_{i_0}, j_0)$. Since $(x_{i_1}, j_1) \in \mathcal{B}$ and $\mathcal{B}$ is all-blue, $j_1 > i_1 > j_0$. Since $\mathcal{B}$ is symmetrically embedded, there is an automorphism $\widetilde{\sigma} \in \mathrm{Aut}(X[\omega]^s)$ extending σ. Define $\tau \in \mathrm{Aut}(X[\omega]^s)$ as follows:

$$\tau(x_i, j) := \begin{cases} (x_{i_1}, j) & \text{if } i = i_0 \\ (x_{i_0}, j) & \text{if } i = i_1 \\ (x_i, j) & \text{if } i \neq i_0, i_1. \end{cases}$$

Namely, τ is the automorphism swapping $\{x_{i_0}\} \times \omega$ and $\{x_{i_1}\} \times \omega$.

Now $\tau \circ \widetilde{\sigma}[\{x_{i_0}\} \times \omega] = \{x_{i_0}\} \times \omega$, so $\tau \circ \widetilde{\sigma} \restriction (\{x_{i_0}\} \times \omega)$ is an automorphism of $\{x_{i_0}\} \times \omega$ and $\tau \circ \widetilde{\sigma}(x_{i_0}, j_0) = (x_{i_0}, j_1)$. This is a non-trivial automorphism of $\{x_{i_0}\} \times \omega$, but $(\{x_{i_0}\} \times \omega) \cong \omega$ is rigid. ☐

Lemma 4.2. Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures and let $\sigma \in \mathrm{Aut}(\mathcal{M})$. If $\widetilde{\sigma} : \mathcal{M}[\mathcal{N}]^s \to \mathcal{M}[\mathcal{N}]^s$ is defined by $\widetilde{\sigma}((a, b)) = (\sigma(a), b)$, then $\widetilde{\sigma}$ is an automorphism.

In particular, if $\mathcal{M}$ is transitive and $A, B \subset \mathcal{M}[\mathcal{N}]^s$ are s-equivalence classes, then there is an automorphism $\tau \in \mathrm{Aut}(\mathcal{M}[\mathcal{N}]^s)$ such that $\tau[A] = B$.

Proof. Clearly $\widetilde{\sigma}$ is a bijection. Notice that $\widetilde{\sigma^{-1}} = (\widetilde{\sigma})^{-1}$, and since σ is arbitrary, proving that $\widetilde{\sigma}$ is a homomorphism will suffice. It is clear that $\widetilde{\sigma}$ preserves s

Let $R \in \mathcal{L}$ be an n-ary relation, $\overline{(a, b)} := ((a_1, b_1)), \ldots, (a_n, b_n)) \in \mathcal{M}[\mathcal{N}]^s$ and assume

$$\mathcal{M}[\mathcal{N}]^s \models R\left(\overline{(a, b)}\right).$$

From the definition of $\mathcal{M}[\mathcal{N}]^s$, one of the following holds:

- $\bigvee_{1\leq j,k\leq n} a_j \neq a_k$ and $\mathcal{M} \models R(a_1, \ldots, a_n)$, so since σ is an automorphism, $\bigvee_{1\leq j,k\leq n} \sigma(a_j) \neq \sigma(a_k)$ and $\mathcal{M} \models R(\sigma(a_1), \ldots, \sigma(a_n))$.
- $\bigwedge_{1\leq j,k\leq n} a_j = a_k$ and $\mathcal{N} \models R(b_1, \ldots, b_n)$, so $\bigwedge_{1\leq j,k\leq n} \sigma(a_j) = \sigma(a_k)$ and $\mathcal{N} \models R(b_1, \ldots, b_n)$.

In any case,

$$\mathcal{M}[\mathcal{N}]^s \models R\left(\widetilde{\sigma}(a_1, b_1), \ldots, \widetilde{\sigma}(a_1, b_1)\right).$$

$\square$

Proposition 4.3. If $\mathcal{M}$ is a transitive structure and $\mathcal{N}$ is a structure with infinitely many orbits such that $\mathcal{N}$ can not be embedded into any finite union of orbits, then $\mathcal{M}[\mathcal{N}]^s$ is not symmetrically indivisible.

Proof. We generalize the proof of Example 4.1: let $\{a_i\}_{i\in\omega}$ be an enumeration of M and $\{O_i\}_{i\in\omega}$ an enumeration of the orbits of $\mathcal{N}$. For $b \in N$, denote $on(b) = j$ if $b \in O_j$ and define a coloring $c : \mathcal{M}[\mathcal{N}]^s \to \{\text{red}, \text{blue}\}$ as follows:

$$c(a_i, b) := \begin{cases} \text{red} & \text{if } on(b) \le i \\ \text{blue} & \text{if } on(b) > i. \end{cases}$$

For every all-red substructure, every s-equivalence class will be embedded in a finite union of orbits, and thus not isomorphic $\mathcal{N}$. It is left to show that there is no all-blue symmetrically embedded substructure isomorphic to $\mathcal{M}[\mathcal{N}]$. Assume towards contradiction $\mathcal{B}$ is such a structure and let $(a_{i_0}, b) \in \mathcal{B}$. Denote $j_0 := on(b)$. Since $\mathcal{B} \cong \mathcal{M}[\mathcal{N}]^s$, $\mathcal{B}$ has infinitely many infinite s-equivalence classes and every s-equivalence class of $\mathcal{B}$ is of the form $\mathcal{B} \cap (\{a\} \times \mathcal{N})$ for some $a \in M$. Let $i_1 > j_0$ such that $\mathcal{B} \cap (\{a_{i_1}\} \times \mathcal{N}) \neq \emptyset$. Since $\mathcal{M}$ is transitive, by Lemma 4.2, for every two s-equivalence classes $A, B \subset \mathcal{M}[\mathcal{N}]^s$, there is an automorphism $\tau \in \mathrm{Aut}(\mathcal{M}[\mathcal{N}]^s)$ such that $\tau[A] = B$. Since $\mathcal{B} \cong \mathcal{M}[\mathcal{N}]^s$, this is true for $\mathcal{B}$ as well, so let $\tau \in \mathrm{Aut}(\mathcal{B})$ be an automorphism such that

$$\tau[\mathcal{B} \cap (\{a_{i_0}\} \times \mathcal{N})] = \mathcal{B} \cap (\{a_{i_1}\} \times \mathcal{N}).$$

Denote $(a_{i_1}, c) := \tau(a_{i_0}, b)$. Since (a_{i_1}, c) is blue, $on(c) > i_1 > j_0 = on(b)$.

Since $\mathcal{B}$ is symmetrically embedded, let $\widehat{\tau} \in \mathrm{Aut}(\mathcal{M}[\mathcal{N}]^s)$ extending τ. Let $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma(a_{i_1}) = a_{i_0}$ and let $\widetilde{\sigma} \in \mathrm{Aut}(\mathcal{M}[\mathcal{N}]^s)$ as defined in Lemma 4.2. $\widetilde{\sigma} \circ \widehat{\tau}$ is an automorphism and $\widetilde{\sigma} \circ \widehat{\tau}[\{a_{i_0}\} \times \mathcal{N}] = \{a_{i_0}\} \times \mathcal{N}$, so

$$\theta := \widetilde{\sigma} \circ \widehat{\tau} \restriction \{a_{i_0}\} \times \mathcal{N}$$

is an automorphism of $\{a_{i_0}\} \times \mathcal{N}$. Define $\iota_0 : \mathcal{N} \xrightarrow{\cong} \{a_{i_0}\} \times \mathcal{N}$ by $\iota_0(b) := (a_{i_0}, b)$. $\iota_0^{-1} \circ \theta \circ \iota$ is an automorphism of $\mathcal{N}$ and $\iota_0^{-1} \circ \theta \circ \iota(b) = c$, but this contradicts $on(c) > on(b)$. $\square$

Theorem 4.4. There is an elementarily indivisible structure that is not symmetrically indivisible.

Proof. Let $\mathcal{A} := \mathcal{M}[\mathcal{N}_a]^s_{a\in M}$ as in Theorem 3.8 and let $\mathcal{B}$ be any elementarily indivisible transitive structure. (The classic examples are all transitive.) If we choose $\{\mathcal{N}_a\}_{a\in M}$ such that $\{\, a \in M \mid \mathcal{A}_i = \mathcal{N}_a \,\}$ is finite for every $i \in \omega$, then by Lemma 3.4 every orbit of $\mathcal{A}$ has only finitely many s-equivalence classes and $\mathcal{A}$ can not be embedded into only finitely many orbits. By Theorem 2.20, $\mathcal{B}[\mathcal{A}]^s$ is elementarily indivisible, but by Proposition 4.3, it is not symmetrically indivisible. $\square$

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